Notes on Diffraction for Physics 410/510, Spring '95

DIFFRACTION

Kirchhoff Scalar Diffraction Theory

The Wave Equation

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

can be solved for the case of monochromatic optical disturbances.

Each Component of $\vec{E}$, $\vec{B}$ will have the form $\psi(\vec{r})\, e^{-i\omega t}$

or $\underbrace{\nabla^2 \psi + k^2 \psi = 0}_{\text{spatial part}}$ Helmholtz Equation

Scalar wave approximation - neglect vector properties - take disturbance to be that of a simple scalar variable ψ with angular frequency ω and wavevector $\vec{k}$

$$|k| = \frac{2\pi}{\lambda} = \frac{\omega}{c}$$

A solution of $\nabla^2 \psi + k^2 \psi = 0$ is spherical coordinates is

$$\psi(r) = \frac{e^{ikr}}{r}$$

corresponds to a spherical wave moving outward in radial direction.

Green's Integral Theorem

From divergence theorem

$$\int_V \operatorname{div} \vec{A}\, dV = \oint_S \vec{A} \cdot \vec{n}\, da \qquad d\vec{S} = \vec{n}\, da$$

Let $\vec{A} = \psi \operatorname{grad} \phi$ (ψ and ϕ are scalar functions)

$$\nabla \cdot (\psi \nabla \phi) = \psi \nabla \cdot (\nabla \phi) + \nabla \phi \cdot \nabla \psi \qquad \nabla \cdot (\nabla \phi) \equiv \nabla^2 \phi$$

$$\oint_S \vec{A} \cdot \vec{n}\, da = \oint_S \psi \frac{d\phi}{dn}\, da = \int_V [\psi \nabla^2 \phi + \nabla \phi \cdot \nabla \psi]\, dV$$

interchange ψ and ϕ and subtract

$$\oint_S \left(\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n}\right) da = \int_V (\psi \nabla^2 \phi - \phi \nabla^2 \psi)\, dV$$

or,

$$\boxed{\oiint_S (\psi_1 \nabla \psi_2 - \psi_2 \nabla \psi_1) \cdot d\vec{S} = \iiint_V (\psi_1 \nabla^2 \psi_2 - \psi_2 \nabla^2 \psi_1)\, dV}$$

The use of an integral equation provides a convenient way to handle boundary conditions. We express desired solution in terms of an integral equation involving a second solution chosen for mathematical convenience.

We can solve the Helmholtz Equation for the diffraction problem with the use of Green's theorem.

Given two solutions of the Helmholtz Eqn.

$$\nabla^2\psi_1 + k^2\psi_1 = 0$$

$$\nabla^2\psi_2 + k^2\psi_2 = 0$$

Green's Integral Theorem $\quad \iiint_V (\psi_1\nabla^2\psi_2 - \psi_2\nabla^2\psi_1)\, dv = \oiint_S (\psi_1\nabla\psi_2 - \psi_2\nabla\psi_1)\cdot d\vec{s}$

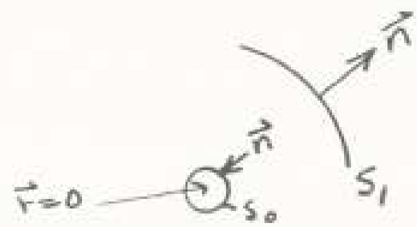
reduces to: $\quad 0 = \oiint_S (\psi_1\nabla\psi_2 - \psi_2\nabla\psi_1)\cdot d\vec{s}$

Now if $\psi_1 = \psi$, the space portion of an arbitrary optical disturbance, and

$$\psi_2 = \frac{e^{ikr}}{r}, \quad \star$$

then $\quad \oiint_{S_1}\left[\psi\,\nabla\left(\frac{e^{ikr}}{r}\right) - \frac{e^{ikr}}{r}\nabla\psi\right]\cdot d\vec{s}_1$

$$+ \oiint_{S_0}\left[\psi\,\nabla\left(\frac{e^{ikr}}{r}\right) - \frac{e^{ikr}}{r}\nabla\psi\right]\cdot d\vec{s}_0 = 0$$



Consider a two-sheet surface excluding $\vec{r}=0$

$\psi_0 = \psi(0)$; $\frac{e^{ikr}}{r}$ is singular at $\vec{r}=0$

$\star \quad \frac{\psi_2 e^{ikr}}{r}$ is a solution of the Helmholtz Eqn. (an outward propagating spherical wave.)

$\psi_1 = \psi$ is the optical disturbance we want to compute

ψ_2 has no physical significance in our problem

$$\psi_2 = \frac{1}{r} e^{ikr}$$

$$\nabla \psi_2 = \frac{\vec{e}_r}{r} ik e^{ikr} - \frac{\vec{e}_r}{r^2} e^{ikr} = -\frac{\vec{e}_r}{r^2} e^{ikr} (1 - ikr)$$

on small shere S_0, $d\vec{S} = r^2 d\Omega (-\vec{e}_r)$ $\quad d\Omega = \sin\theta d\theta d\phi$

$$\oiint_{S_0} \left(\psi \nabla \left(\frac{e^{ikr}}{r} \right) - \frac{e^{ikr}}{r} \nabla \psi \right) \cdot d\vec{S}$$

$$= \oiint_{S_0} \left(\psi - ikr\psi + r \frac{\partial \psi}{\partial r} \right) e^{ikr} d\Omega$$

shrink small sphere

continuity of $\psi \Rightarrow$ value of ψ on $S_0 \rightarrow \psi_0 = \psi(0)$

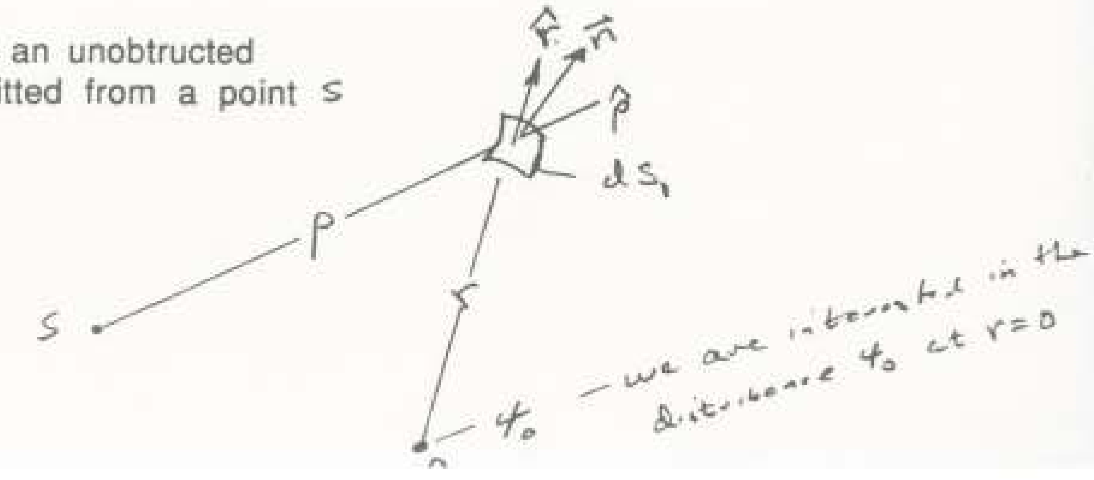
$e^{ikr} \rightarrow 1$

and

$$\oiint_{S_0} \rightarrow 4\pi \psi_0$$

Thus, $\psi_0 = \frac{1}{4\pi} \left[\oiint_S \frac{e^{ikr}}{r} \nabla \psi \cdot d\vec{S} - \oiint_S \psi \nabla \left(\frac{e^{ikr}}{r} \right) \cdot d\vec{S} \right]$

Now consider an unobtructed spherical wave emitted from a point S



$$\psi(\vec{\rho}) = \frac{A}{\rho} e^{ik\rho}$$

$$\psi_o = \frac{1}{4\pi}\Big[\iint_{S_1} \frac{e^{ikr}}{r} \frac{\partial}{\partial \rho}\left(\frac{A}{\rho} e^{ik\rho}\right) \cos(\hat{n},\hat{\rho})\, dS_1$$

$$- \iint_{S_1} \frac{A}{\rho} e^{ik\rho} \frac{\partial}{\partial r}\left(\frac{e^{ikr}}{r}\right) \cos(\hat{n},\hat{r})\, dS_1\Big]$$

$$\frac{\partial}{\partial \rho} \frac{e^{ik\rho}}{\rho} = e^{ik\rho}\left(\frac{ik}{\rho} - \frac{1}{\rho^2}\right)$$

$$\frac{\partial}{\partial r} \frac{e^{ikr}}{r} = e^{ikr}\left(\frac{ik}{r} - \frac{1}{r^2}\right)$$

Now if $\rho \gg \lambda$ and $r \gg \lambda$, we can neglect

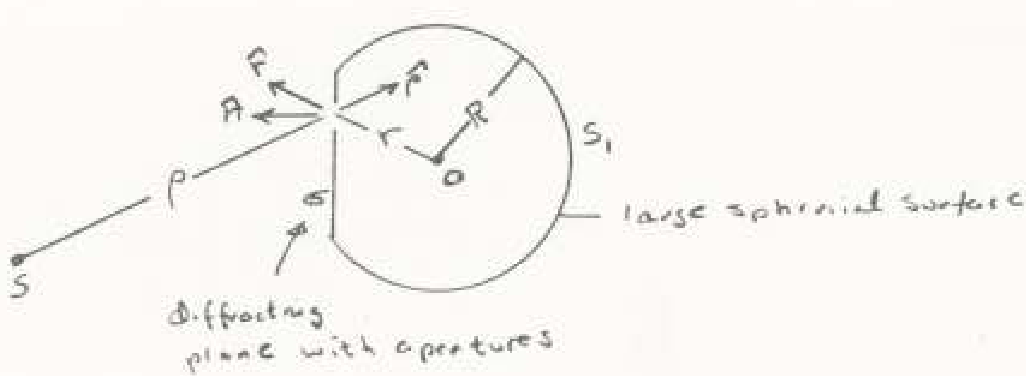
$\frac{1}{\rho^2}$ and $\frac{1}{r^2}$ terms $\quad k = \frac{2\pi}{\lambda} \gg \frac{1}{\rho}, \frac{1}{r}$

good approximation at optical wavelengths; this gives finally

$$\boxed{\psi_o = -\frac{Ai}{\lambda} \iint_{S_1} \frac{e^{ik(\rho+r)}}{\rho r} \left[\frac{\cos(\hat{n},\hat{r}) - \cos(\hat{n},\hat{\rho})}{2}\right] dS_1}$$

Fresnel-Kirchhoff Diffraction Formula

Now consider the geometry below:



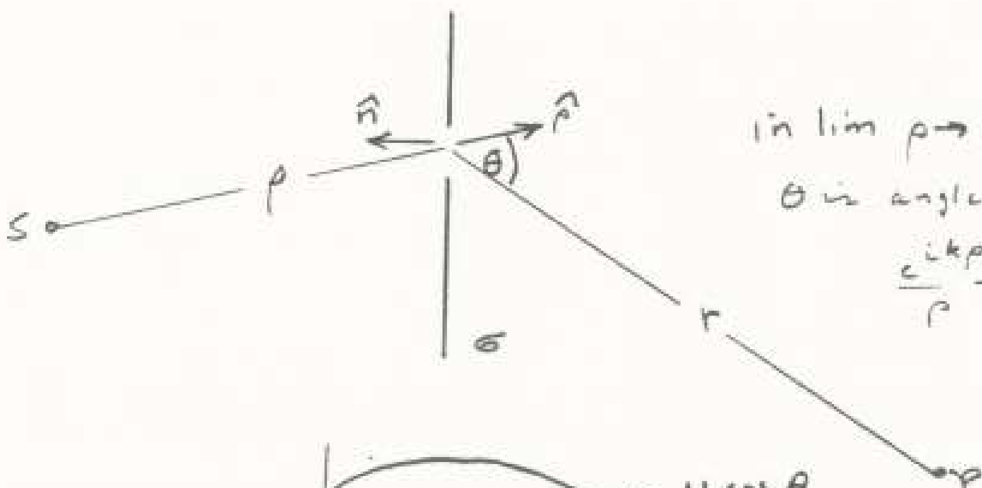
As $R \to \infty$ integral over the large spherical surface vanishes*

If source at s is located far from the diffracting screen, we have a plane wave incident on the diffracting screen and $\hat{n}$ and $\hat{\rho}$ are antiparallel

i.e., $\cos(\hat{n},\hat{\rho}) = -1$ and $\cos(\hat{n},\hat{r}) = \cos\theta$

amplitude at arbitrary point P (at origin of previous figure)

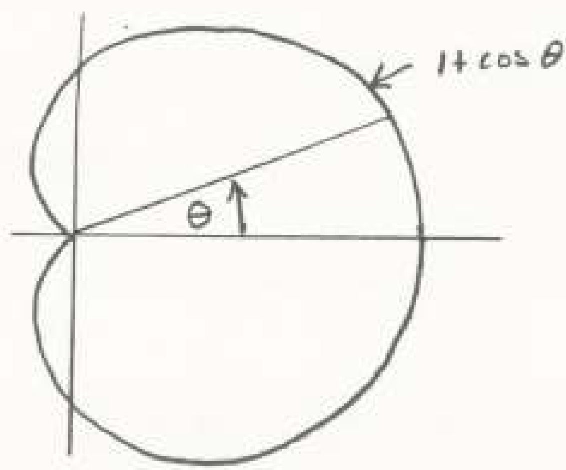
$$\psi(P) = \frac{-i}{2\lambda}\underbrace{\left[\frac{Ae^{ik\rho}}{\rho}\right]}_{\text{spherical wave from S}}\iint_\sigma \frac{e^{ikr}}{r}(1+\cos\theta)\,d\sigma$$



in lim $\rho \to \infty$

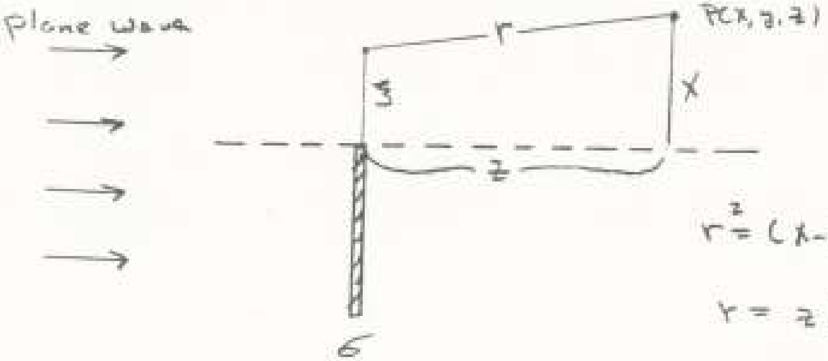
θ is angle to normal

$\frac{e^{ik\rho}}{\rho} \to$ plane wave



*see Born and Wolf, Principles of Optics, p. 379.

Diffraction by an edge - the edge wave



$$r^2 = (x-\xi)^2 + (y-\eta)^2 + z^2$$

$$r = z\left[1 + \frac{(x-\xi)^2}{z^2} + \frac{(y-\eta)^2}{z^2}\right]^{1/2}$$

$$\approx z\left[1 + \frac{(x-\xi)^2}{2z^2} + \frac{(y-\eta)^2}{2z^2} + \cdots\right]$$

if we assume $|x-\xi| \ll z$
$|y-\eta| \ll z$

Thus $r \approx z + \frac{(x-\xi)^2}{2z} + \frac{(y-\eta)^2}{2z}$

Take $\cos\theta \approx 1$

plane wave amplitude at $z=0$

$$\psi(P) = -\frac{iA}{\lambda}\int_\sigma \frac{e^{ikr}}{r}\, dS = -\frac{iA}{\lambda}\frac{e^{ikz}}{z}\int_\sigma \exp \frac{ik}{2z}\left[(x-\xi)^2 + (y-\eta)^2\right] d\xi\, d\eta$$

use $r \approx z$ in denominator

i.e. $\frac{1}{r} = \frac{1}{z}\left[1 - \frac{(x-\xi)^2}{2z^2} - \frac{(y-\eta)^2}{2z^2} + \cdots\right]$

negligible contribution to amplitude, but important in phase

let $\frac{\pi}{2}u^2 = \frac{k}{2z}(x-\xi)^2$

$$x - \xi = \sqrt{\frac{\pi z}{k}}\, u$$

$$d\xi = -\sqrt{\frac{\pi z}{k}}\, du \quad \text{(at fixed value of } z\text{)}$$

$$\text{at } \xi = 0,\ u(\xi=0) = \sqrt{\frac{k}{\pi z}}\, x = u_0$$

$$\psi(P) = -\frac{iA}{\lambda z}e^{ikz}\left(-\sqrt{\frac{\pi z}{k}}\right)\left[-\int_{-\infty}^{u_0} \exp\left[\frac{i\pi}{2}w^2\right] du\right]\left[\int_{-\infty}^{\infty} \exp\left[\frac{ik}{2z}(y-\eta)^2\right] d\eta\right]$$

restrict to $y=0$ case,

$$\int_{-\infty}^{\infty} e^{\frac{ik}{2z}\eta^2}\, d\eta = \sqrt{\frac{\pi z}{k}}\int_{-\infty}^{\infty} e^{\frac{i\pi}{2}w^2}\, dw \qquad \frac{k}{2z}\eta^2 = \frac{\pi}{2}w^2$$

$$= \sqrt{\frac{\pi z}{k}} \left[\int_{-\infty}^{\infty} \cos \frac{\pi}{2} \omega^2 d\omega + i \int_{-\infty}^{\infty} \sin \frac{\pi}{2} \omega^2 d\omega \right]$$

$$= \sqrt{\frac{\pi z}{k}} (1+i)$$

$-i(1+i) = 1-i$, thus

$$\psi(P) = \frac{(1-i)A}{2} e^{ikz} \left[\int_{-\infty}^{u_0} e^{i\frac{\pi}{2}u^2} du \right] \qquad \int_{-\infty}^{\infty} e^{\frac{i\pi}{2}u^2} du = 1+i$$

For the incident wave we have $\psi = \psi_0$ (calculate with edge removed), i.e.,

$$\psi_0 = \frac{(1-i)A}{2} e^{ikz} (1+i) = A e^{ikz}$$

$$|\psi_0|^2 = I_0 = A^2$$

$$|\psi(P)|^2 = \left| \frac{(1-i)A}{2} e^{ikz} \right|^2 \left| \left[\int_{-\infty}^{u_0} e^{i\frac{\pi}{2}u^2} du \right] \right|^2$$

$$= \frac{1}{2} |A|^2 \left| \left[\int_{-\infty}^{u_0} e^{i\frac{\pi}{2}u^2} du \right] \right|^2$$

$$I(P) = |\psi(P)|^2 = \frac{1}{2} I_0 \left| \int_{-\infty}^{u_0} e^{i\frac{\pi}{2}u^2} du \right|^2$$

$$= \frac{I_0}{2} \left| \int_{-\infty}^{u_0} \cos \frac{\pi}{2} u^2 du + i \int_{-\infty}^{u_0} \sin \frac{\pi}{2} u^2 du \right|^2$$

$$= \frac{I_0}{2} \left\{ \left[\int_{-\infty}^{u_0} \cos \frac{\pi}{2} u^2 du \right]^2 + \left[\int_{-\infty}^{u_0} \sin \frac{\pi}{2} u^2 du \right]^2 \right\}$$

Define

$$C(u) = \int_0^u \cos\frac{\pi}{2}u^2\,du \qquad \int_{-\infty}^{\infty} e^{i\frac{\pi}{2}u^2}\,du = 1+i$$

$$S(u) = \int_0^u \sin\frac{\pi}{2}u^2\,du$$

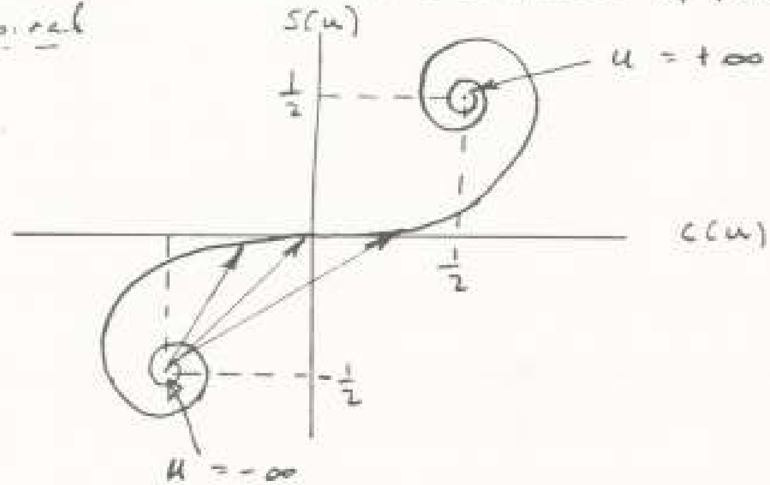
$$C(0) = 0 \qquad S(0) = 0$$

$$C(-u) = -C(u) \qquad S(-u) = -S(u)$$

$$C(\infty) = \frac{1}{2} \qquad S(\infty) = \frac{1}{2}$$

$$C(-\infty) = -\frac{1}{2} \qquad S(-\infty) = -\frac{1}{2}$$

Let C(u) and S(u) be the rectangular coordinates of a point Q(u). As u takes on all values from $-\infty < u < \infty$, the locus of points is the Cornu Spiral



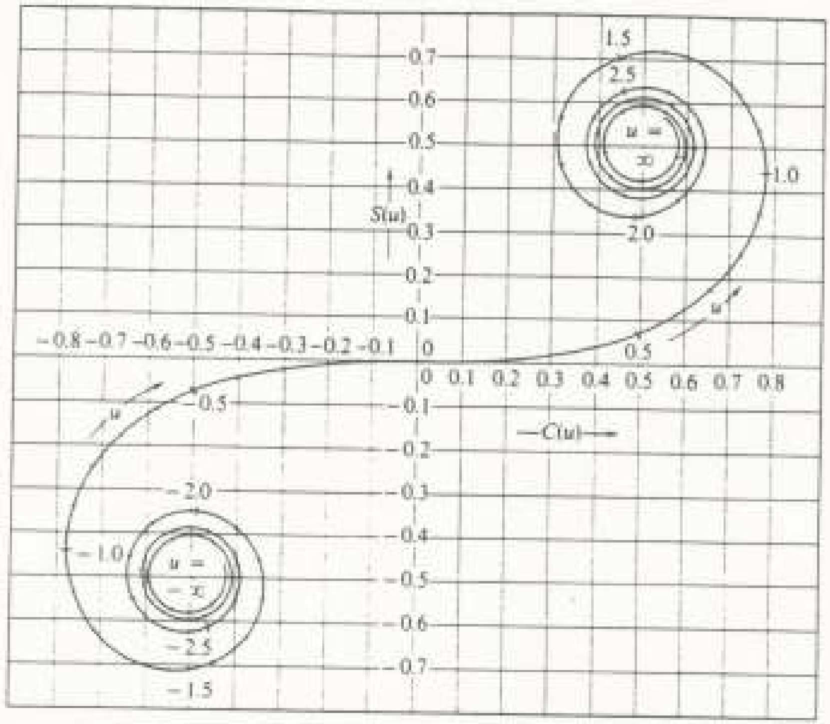
The length L of the straight line from any point $Q(u_1)$ on the Cornu Spiral to any other point $Q(u_2)$ is

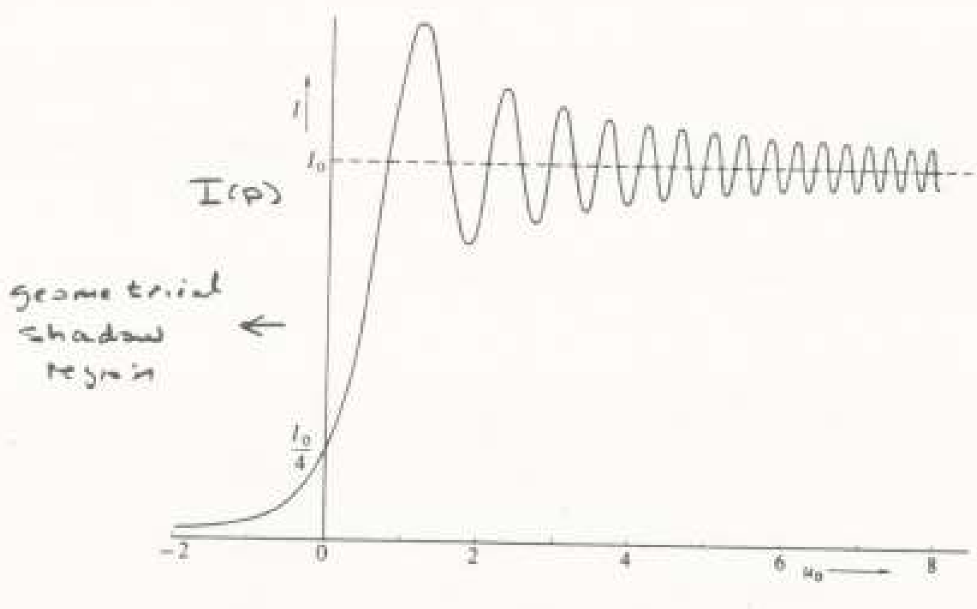
$$L^2 = \left[C(u_2) - C(u_1)\right]^2 + \left[S(u_2) - S(u_1)\right]^2$$

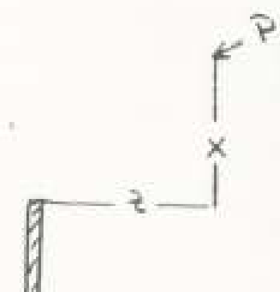
$I(p)$ is proportional to the square of the length of the straight line from $Q(-\infty)$ to $Q(u_0)$, i.e.,

$$I(p) = \frac{I_0}{2}\left\{\left[C(u_0) - C(-\infty)\right]^2 + \left[S(u_0) - S(-\infty)\right]^2\right\}$$

The Cornu Spiral





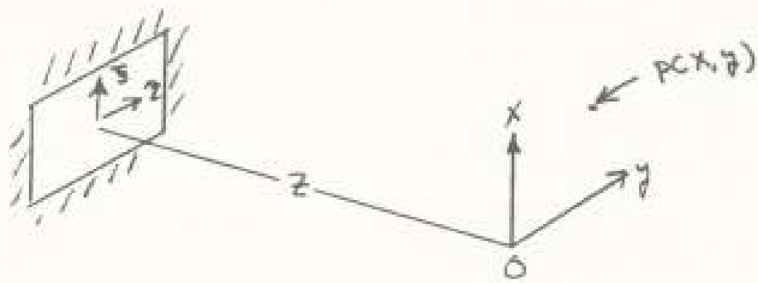


$$u_0 \equiv \sqrt{\frac{k}{\pi z}}\, x$$

Diffraction by an edge

The Rectangular Aperture

$$\psi_p = -\frac{iA}{\lambda}\,\frac{e^{ikz}}{z}\int_\sigma \exp \frac{ik}{2z}\left[(x-\xi)^2 + (y-\eta)^2\right] d\xi\, d\eta$$



Let $\frac{\pi}{2}u^2 = \frac{k}{2z}(x-\xi)^2$ $\qquad d\xi = -\sqrt{\frac{\pi z}{k}}\,du$

$\frac{\pi}{2}v^2 = \frac{k}{2z}(y-\eta)^2$ $\qquad d\eta = -\sqrt{\frac{\pi z}{k}}\,dv$

$$\psi_p = -\frac{iA}{2} e^{ikz}\left[\int_{u_1}^{u_2} \exp\left[i\tfrac{\pi}{2}u^2\right] du\right]\left[\int_{v_1}^{v_2} \exp\left[i\tfrac{\pi}{2}v^2\right] dv\right]$$

$$= -\frac{iA}{2} e^{ikz}\left[C(u) + iS(u)\right]_{u_1}^{u_2}\left[C(v) + iS(v)\right]_{v_1}^{v_2}$$

$$\psi_0 = -\frac{iA}{2} e^{ikz}(1+i)(1+i) = A e^{ikz}$$

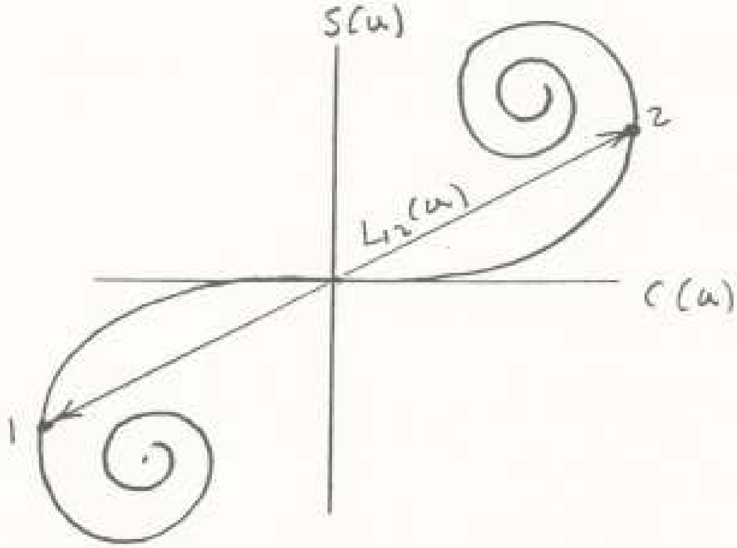
$I_0 = |\psi_0|^2 = A^2$ (as before)

$$\frac{I_p}{I_0} = \frac{|\psi_p|^2}{|\psi_0|^2} \Rightarrow I_p = \frac{I_0}{4}\left\{\left[C(u_2) - C(u_1)\right]^2 + \left[S(u_2) - S(u_1)\right]^2\right\}$$
$$\times\left\{\left[C(v_2) - C(v_1)\right]^2 + \left[S(v_2) - S(v_1)\right]^2\right\}$$

$$I_p = \frac{I_0}{4} L_{12}^2(u)\, L_{12}^2(v)$$

If $x, y = 0$ and a square aperture

$$I_p = \frac{I_0}{4} L_{12}^4(u)$$



If $x, y = 0$ and $-\infty < \eta < \infty$) (infinite slit)

$$I_p = \frac{I_0}{2} L_{12}^2(u)$$